Singular solutions of fractional elliptic equations with absorption

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Abstract

The aim of this paper is to study the singular solutions to fractional elliptic equations with absorption

$$\begin{cases} (-\Delta)^\alpha u + |u|^{p-1}u = 0, & \text{in} \quad \Omega \setminus \{0\}, \\ u = 0, & \text{in} \quad \mathbb{R}^N \setminus \Omega, \\ \lim_{x\to 0} u(x) = +\infty, \end{cases}$$

where $p > 0$, Ω is an open, bounded and smooth domain of $\mathbb{R}^N$ ($N \geq 2$) with $0 \in \Omega$.

We analyze the existence, nonexistence, uniqueness and asymptotic behavior of the solutions.

1 Introduction

In the present paper, we are concerned with the singular solutions of fractional elliptic problems of the form

$$\begin{cases} (-\Delta)^\alpha u + |u|^{p-1}u = 0, & \text{in} \quad \Omega \setminus \{0\}, \\ u = 0, & \text{in} \quad \mathbb{R}^N \setminus \Omega, \\ \lim_{x\to 0} u(x) = +\infty, \end{cases} \tag{1.1}$$

where Ω is an open, bounded and smooth domain of $\mathbb{R}^N$ $(N \geq 2)$ with $0 \in \Omega$, $p > 0$ and $(-\Delta)^\alpha$ with $\alpha \in (0,1)$ is the fractional Laplacian defined as

$$(-\Delta)^\alpha u(x) = P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2\alpha}} dy, \tag{1.2}$$

here $P.V.$ stands for the principle value integral, that for notational simplicity we omit in what follows.

During the last years, singular solutions of nonlinear elliptic equations have been studied by many authors. We just mention the earlier work by Véron [29, 30], Gmira-Véron [24], Brezis-Lions [7], Bandle-Marcus [2, 3], Baras-Pierre [4], Chen-Matano-Véron [21], without any attempt to review the references here. The first result of unconditional removability of isolated sets for semilinear elliptic equations with absorption term is due to Brezis-Véron [8]. They considered the classical equation

$$-\Delta u + g(u) = 0 \quad \text{in} \quad \Omega \setminus \{0\}, \tag{1.3}$$

where Ω is an open subset of $\mathbb{R}^N$ $(N \geq 3)$ containing 0 and g is a continuous function satisfying some extra hypothesis, then there exists a solution for equation (1.3) in the whole Ω. Later on, this result was extended in [30] using the method which is developed by Baras-Pierre [4]. In the meantime, Véron [29] has done much work for equation (1.3) with $g(u) = |u|^{p-1}u$ and $1 < p < \frac{N}{N-2}$ if $N \geq 3$ $(p > 1 \; if \; N = 2)$, he described the behaviour of solution for equation (1.3) near the isolated singularity.

Recently, great attention has been devoted to investigate nonlinear equations involving fractional Laplacian. Caffarelli-Silvestre [20] gave a new formulation of the fractional Laplacian through Dirichlet-Neumann maps. Later, they studied the regularity results for fractional problems in [13, 14]. The existence of solution for equation with fractional Laplacian was proved by Cabré-Tan [10], Felmer-Quaas [22], Servadei-Valdinoci [27]. Moreover, Li [25], Chen-Li-Ou [16, 17] and Felmer-Wang [23] studied symmetry results and monotonicity of positive solutions for fractional equations. Chen-Felmer-Quaas [18] analyzed the existence and asymptotic behavior of large solution to fractional equation with absorption by advanced method of super and sub solutions.

The purpose of this paper is to study singular solutions for fractional equations (1.1) with absorption, including the existence and the asymptotic behavior of singular solutions near 0. It is well-known that the singular near 0 of functions $|x|^\tau$ only could be considered with $\tau \in (-N, 0)$ for working by fractional laplacian, which is a nonlocal operator. In the following, we state main result.

Theorem 1.1 *Suppose that Ω is an open, bounded and smooth domain of $\mathbb{R}^N$ $(N \geq 2)$ with $0 \in \Omega$, $\alpha \in (0,1)$.*

(i) If

$$1 + \frac{2\alpha}{N} < p < \frac{N}{N-2\alpha}, \tag{1.4}$$

then problem (1.1) admits a positive solution u such that for some $C > 0$,

$$\lim_{x\to 0} u(x)|x|^{\frac{2\alpha}{p-1}} = C. \tag{1.5}$$

Moreover, that solution u is unique in the sense of

$$0 < \liminf_{x\to 0} u(x)|x|^{\frac{2\alpha}{p-1}} \leq \limsup_{x\to 0} u(x)|x|^{\frac{2\alpha}{p-1}} < +\infty. \tag{1.6}$$

(ii) If

$$0 < p < \frac{N}{N-2\alpha}, \tag{1.7}$$

then for any $t > 0$, problem (1.1) admits a positive solution u such that

$$\lim_{x\to 0} u(x)|x|^{N-2\alpha} = t. \tag{1.8}$$

(iii) If $p > 0$, then problem (1.1) doesn't admit any solution u such that

$$0 < \liminf_{x\to 0} u(x)|x|^{-\tau} \leq \limsup_{x\to 0} u(x)|x|^{-\tau} < +\infty, \tag{1.9}$$

for any $\tau \in (-N, 0) \setminus \{2\alpha - N, -\frac{2\alpha}{p-1}\}$.

Theorem 1.1 part (i) presents the existence, uniqueness in the sense of (1.6) and the asymptotic behavior with power $-\frac{2\alpha}{p-1}$ of singular solution to (1.1), part (ii) shows the existence and the asymptotic behavior with power $-N + 2\alpha$ of singular solution to (1.1) and part (iii) gives the nonexistence of singular solution to (1.1) in the sense (1.9). In the next, we give some remarks to show more information for singular solution to (1.1).

Remark 1.1 *Under the hypothesis of Theorem 1.1 part (i), the solution u, which satisfies (1.5), has estimate*

$$|u(x) - C_1|x|^{-\frac{2\alpha}{p-1}}| < C_2, \quad x \in \Omega \setminus \{0\} \tag{1.10}$$

where $C_1 > 0$ will be given in (3.1) and $C_2 > 0$.

Remark 1.2 *Under the hypothesis of Theorem 1.1 part* (ii)*, if*

$$\frac{2\alpha}{N-2\alpha} < p < \frac{N}{N-2\alpha}, \tag{1.11}$$

then for any $t > 0$*, problem (1.1) admits a positive solution* u *such that, for any* $0 < |x| < d_0$*, we have*

$$\frac{|x|^{\tau_1}}{C} \le t|x|^{2\alpha-N} - u(x) \le C|x|^{\tau_1}, \tag{1.12}$$

where $C > 0$*,* $\tau_1 = 2\alpha - (N-2\alpha)p < 0$ *and* $d_0 = \frac{1}{3}\min\{dist(0, \partial\Omega), 1\}$*.*

Remark 1.3 *Under the hypothesis of Theorem 1.1 part* (iii)*, if* $p \ge \frac{N}{N-2\alpha}$*, then problem (1.1) doesn't admit any solution* u *such that*

$$0 < \liminf_{x\to 0} u(x)|x|^{-\tau} \le \limsup_{x\to 0} u(x)|x|^{-\tau} < +\infty, \tag{1.13}$$

for any $\tau \in (-N, 0)$*.*

The rest of the paper is organized as follows. In Section §2, we introduce Preliminaries for existence and some estimates which is used for constructing super and sub solutions of (1.1). In Section §3, we prove the existence of the solutions of (1.1). The uniqueness is addressed in Section §4. In the section §5, it is devoted to non-existence.

2 Preliminaries

We remind here some basic knowledge about $(-\Delta)^\alpha$ with $\alpha \in (0, 1)$, see for instance [18].

Lemma 2.1 *Assume that* x_0 *achieves the maximum of* u *in* $\mathbb{R}^N$*, then*

$$(-\Delta)^\alpha u(x_0) \ge 0. \tag{2.1}$$

Moreover, if x_0 *achieves the maximum of* u *in* $\mathbb{R}^N$*, then*

$$(-\Delta)^\alpha u(x_0) \le 0, \tag{2.2}$$

holds if and only if

$$u(x) = u(x_0) \quad \text{a.e. in} \quad \mathbb{R}^{\mathrm{N}}.$$

Lemma 2.2 *Assume that $0 \in \Omega$ and $p > 0$. Moreover, we suppose that there are super-solution $\bar{U}$ and sub-solution $\underline{U}$ of (1.1) such that*

$$\bar{U} \geq \underline{U} \ \text{ in } \Omega \setminus \{0\}, \quad \liminf_{\mathrm{x} \to 0} \underline{\mathrm{U}}(\mathrm{x}) = +\infty, \quad \bar{\mathrm{U}} = \underline{\mathrm{U}} = 0 \ \text{ in } \Omega^{\mathrm{c}}.$$

Then there exists at least one positive solution u of (1.1) such that

$$\underline{U} \leq u \leq \bar{U} \ \text{ in } \Omega \setminus \{0\}.$$

Proof. The process of the proof is the same as Theorem 2.6 in [18]. $\Box$

In order to construct super and sub solutions for problem (1.1), we will use some appropriate truncated functions. To describe our following analysis, we give some notations. By $0 \in \Omega$, it is able to assume that $\delta \in (0, d_0)$ is such that $d(\cdot) = dist(\cdot, \partial\Omega)$ is C^2 in $A_\delta := \{x \in \Omega \mid d(x) < \delta\}$ and $d(x) \leq |x|$ in A_δ, where $d_0 = \frac{1}{3}\min\{dist(0, \partial\Omega), 1\}$. Let $B_r := B_r(0) \setminus \{0\}$ for any $r > 0$, we have $dist(A_\delta, B_{d_0}) > 0$. Moreover, we define

$$V_\tau(x) := \begin{cases} |x|^\tau, & x \in B_{d_0}, \\ d(x)^2, & x \in A_\delta, \\ l(x), & x \in \Omega \setminus (A_\delta \cup B_{d_0}(0)), \\ 0, & x \in \Omega^c, \end{cases} \tag{2.3}$$

where τ is a parameter in $(-N, 0)$ and the function l is positive such that $l(x) \leq |x|^\tau$ in $\Omega \setminus (A_\delta \cup B_{d_0}(0))$ and V_τ is C^2 in $\mathbb{R}^N \setminus \{0\}$.

It will be convenient for next auxiliary lemmas to define the following function

$$C(\tau) := \int_{\mathbb{R}^N} \frac{|z - e_1|^\tau - 1}{|z|^{N+2\alpha}} dz \tag{2.4}$$

where $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^N$. It is well known from [22] that

$$C(\tau) \begin{cases} > 0, & \text{if } \ \tau \in (-\mathrm{N}, -\mathrm{N} + 2\alpha), \\ = 0, & \text{if } \ \tau = -\mathrm{N} + 2\alpha, \\ < 0, & \text{if } \ \tau \in (-\mathrm{N} + 2\alpha, 0). \end{cases} \tag{2.5}$$

Lemma 2.3 *Assume that Ω is an open, bounded, smooth domain with $0 \in \Omega$ and $\tau \in (-N, 0)$. Then there exists $c > 0$ such htat*

$$-c < (-\Delta)^\alpha V_\tau(x) + C(\tau)|x|^{\tau - 2\alpha} \leq 0, \quad \forall\, x \in B_{d_0/2}, \tag{2.6}$$

where $C(\cdot)$ is defined in (2.4).

Proof. For any given $x \in B_{d_0/2}$, we have

$$\begin{aligned}
-(-\Delta)^\alpha V_\tau(x) &= \int_{\mathbb{R}^N} \frac{V_\tau(z) - V_\tau(x)}{|z-x|^{N+2\alpha}} dz = \int_{\mathbb{R}^N} \frac{V_\tau(z) - |x|^\tau}{|z-x|^{N+2\alpha}} dz \\
&= \int_{\mathbb{R}^N} \frac{|z|^\tau - |x|^\tau}{|z-x|^{N+2\alpha}} dz + \int_{\mathbb{R}^N \setminus B_{d_0}} \frac{V_\tau(z) - |z|^\tau}{|z-x|^{N+2\alpha}} dz \\
&=: I_1(x) + I_2(x).
\end{aligned}$$

We look at each of these integrals separately. On one side, by direct computation, we have

$$I_1(x) = \int_{\mathbb{R}^N} \frac{|z+x|^\tau - |x|^\tau}{|z|^{N+2\alpha}} dz = C(\tau)|x|^{\tau - 2\alpha}.$$

On the other side, for $z \in \mathbb{R}^N \setminus B_{d_0}$ and $x \in B_{d_0/2}$, we have $|z - x| \geq \frac{|z|}{2}$ and $|V_\tau(z) - |z|^\tau| \leq c|z|^\tau$ for some $c > 0$. Then there exists $C > 0$ such that

$$\begin{aligned}
I_2(x) &= \int_{\mathbb{R}^N \setminus B_{d_0}} \frac{V_\tau(z) - |z|^\tau}{|z-x|^{N+2\alpha}} dz \\
&\geq -C \int_{\mathbb{R}^N \setminus B_{d_0}} |z|^{\tau - N - 2\alpha} dz \\
&\geq -C d_0^{\tau - 2\alpha}.
\end{aligned}$$

On the other hand by $V_\tau(z) \leq |z|^\tau$, we have

$$I_2(x) = \int_{\mathbb{R}^N \setminus B_{d_0}} \frac{V_\tau(z) - |z|^\tau}{|z-x|^{N+2\alpha}} dz < 0.$$

Hence, we obtain (2.6). The proof is compete. $\square$

As a consequence, we have the following corollary

Corollary 2.1 *Let Ω be an open, bounded, smooth domain containing* 0.
(i) If

$$\tau \in (-N, -N + 2\alpha),$$

then there exists $\delta_1 \in (0, d_0)$ and $C > 1$ such that

$$\frac{1}{C}|x|^{\tau - 2\alpha} \leq -(-\Delta)^\alpha V_\tau(x) \leq C|x|^{\tau - 2\alpha}, \quad \forall\, x \in B_{\delta_1}.$$

(ii) If

$$\tau \in (-N + 2\alpha, 0),$$

then there exists $\delta_1 \in (0, d_0)$ *and* $C > 1$ *such that*

$$\frac{1}{C}|x|^{\tau-2\alpha} \le (-\Delta)^\alpha V_\tau(x) \le C|x|^{\tau-2\alpha}, \quad \forall\, x \in B_{\delta_1}.$$

(iii) If

$$\tau = -N + 2\alpha,$$

then there exists $C > 1$ *such that*

$$|(-\Delta)^\alpha V_\tau(x)| \le C, \quad \forall\, x \in \Omega \setminus \{0\}.$$

Proof. It follows directly Lemma 2.3 and (2.5). $\Box$

3 Existence of Problem (1.1)

This section is devoted to use Corollary 2.1 to to construct suitable sub-solution and super-solution of (1.1) to prove the existence.

Proof of Remark 1.1. Firstly, we construct super-solution and sub-solution of (1.1) under the hypotheses of Theorem 1.1 part (i) by adjusting the parameter $\lambda > 0$ in the following functions

$$U_\lambda(x) := C(\tau_p)^{\frac{1}{p-1}} V_{\tau_p}(x) + \lambda \bar{V}(x) \quad \text{and} \quad \mathrm{W}_\lambda(\mathrm{x}) := \mathrm{C}(\tau_\mathrm{p})^{\frac{1}{\mathrm{p}-1}} \mathrm{V}_{\tau_\mathrm{p}}(\mathrm{x}) - \lambda \bar{\mathrm{V}}(\mathrm{x}), \tag{3.1}$$

where V_{τ_p} is defined in (2.3) with $\tau_p = -\frac{2\alpha}{p-1} \in (-N, -N+2\alpha)$, $C(\tau_p) > 0$ is defined in (2.4) and $\bar{V}$ is the solution of

$$\begin{cases} (-\Delta)^\alpha \bar{V}(x) = 1, & x \in \Omega, \\ \bar{V}(x) = 0, & x \in \Omega^c. \end{cases} \tag{3.2}$$

By Lemma 2.1, we have that $\bar{V} > 0$ in Ω.

1. Super-solution. By the definition of U_λ, it has

$$(-\Delta)^\alpha U_\lambda(x) = C(\tau_p)^{\frac{1}{p-1}} (-\Delta)^\alpha V_{\tau_p}(x) + \lambda \quad \text{in } \Omega \setminus \{0\}.$$

By (2.6) and $\tau_p p = \tau_p - 2\alpha$, it follows that for all $\lambda \ge 0$,

$$(-\Delta)^\alpha U_\lambda(x) + U_\lambda^p(x) \ge -C(\tau_p)^{\frac{p}{p-1}} |x|^{\tau_p - 2\alpha} + C(\tau_p)^{\frac{p}{p-1}} |x|^{\tau_p p} \ge 0, \quad x \in B_{\frac{d_0}{2}}.$$

In above inequality we used that for any $a, b \ge 0$,

$$(a+b)^p \ge a^p.$$

Next we consider the domain $\Omega \setminus B_{\frac{d_0}{2}}(0)$. Then, by definition of V_τ, there exists $C_1 > 0$ such that

$$|(-\Delta)^\alpha V_\tau| \leq C_1 \ \text{ in } \ \Omega \setminus \mathrm{B}_{\frac{\mathrm{d}_0}{2}}(0).$$

Then there exists $\bar\lambda > 0$ such that for $\lambda \geq \bar\lambda$, it has

$$(-\Delta)^\alpha U_\lambda(x) + U_\lambda^p(x) \ \geq \ \lambda - C_1 C(\tau_p)^{\frac{1}{p-1}} \geq 0.$$

Together with $U_{\bar\lambda} = 0$ in Ω^c, we have that $U_{\bar\lambda}$ is a super-solution of (1.1).

2. Sub-solution. We observe that

$$(-\Delta)^\alpha W_\lambda(x) = C(\tau_p)^{\frac{1}{p-1}}(-\Delta)^\alpha V_{\tau_p}(x) - \lambda \quad \text{in } \Omega \setminus \{0\}.$$

By (2.6), it follows that for $x \in B_{\frac{d_0}{2}}$ and $\lambda \geq 0$,

$$(-\Delta)^\alpha W_\lambda(x) + |W_\lambda|^{p-1} W_\lambda(x) \leq -C(\tau_p)^{\frac{p}{p-1}}|x|^{\tau_p - 2\alpha} + C(\tau_p)^{\frac{p}{p-1}}|x|^{\tau_p p} \leq 0.$$

In the first inequality above we used that for any $a, b \geq 0$,

$$|a-b|^{p-1}(a-b) \leq a^p.$$

Since $(-\Delta)^\alpha V_\tau + V_\tau^p$ is continuous in $\Omega \setminus \{0\}$, then there exists $C_2 > 0$ such that

$$|C(\tau_p)^{\frac{1}{p-1}}(-\Delta)^\alpha V_\tau| + C(\tau_p)^{\frac{p}{p-1}} V_\tau^p \leq C_2, \quad x \in \Omega \setminus B_{\frac{d_0}{2}}(0).$$

Then, there exists $\underline{\lambda} > 0$ such that for $\lambda \geq \underline{\lambda}$, we have

$$\begin{aligned} (-\Delta)^\alpha W_\lambda(x) + |W_\lambda|^{p-1} W_\lambda(x) \ &\leq \ C_2 - \lambda \\ &\leq \ 0, \quad x \in \Omega \setminus B_{\frac{d_0}{2}}(0). \end{aligned}$$

Then $W_{\underline{\lambda}}$ is a sub-solution of (1.1). Since $\bar\lambda, \underline{\lambda} > 0$ and $\bar V > 0$ in Ω, then

$$U_{\bar\lambda} > W_{\underline{\lambda}} \ \text{ in } \Omega \setminus \{0\} \ \text{ and } \ \mathrm{U}_{\bar\lambda} = \mathrm{W}_{\underline{\lambda}} = 0 \text{ in } \Omega^{\mathrm{c}}. \tag{3.3}$$

Then, by Lemma 2.2, there exists at least one positive solution u such that

$$W_{\underline{\lambda}} \leq u \leq U_{\bar\lambda} \quad \text{in } \Omega \setminus \{0\}.$$

The proof is complete. $\square$

The proof of Theorem 1.1 part (i) follows the proof of Remark 1.1.

Proof of Theorem 1.1 part (ii) **with** $0 < p \le \frac{2\alpha}{N-2\alpha}$**.** Let $\tau_0 = 2\alpha - N$ and $\tau_1 = \frac{2\alpha - N}{2} < 0$. For $0 < p \le \frac{2\alpha}{N-2\alpha}$, we have that

$$0 > p\tau_0 \ge \tau_1 - 2\alpha.$$

For any given $t > 0$, we define

$$U_\mu(x) := tV_{\tau_0}(x) + \mu\bar{V}(x)$$

and

$$W_\mu(x) := tV_{\tau_0}(x) - \mu V_{\tau_1}(x) - \mu^2\bar{V}(x),$$

where $\mu, \lambda > 0$, V_τ is defined in (2.3) and $\bar{V}$ is the solution of (3.2). We construct super-solution and sub-solution of (1.1) under the hypotheses of Theorem 1.1 part (ii) by adjusting the positive parameters μ.

1. Super-solution. By the definition of U_μ, it has

$$(-\Delta)^\alpha U_\mu(x) = t(-\Delta)^\alpha V_{\tau_0}(x) + \mu, \quad x \in \Omega \setminus \{0\}.$$

By Corollary 2.1 part (iii), for $x \in B_{d_0}$, it follows that

$$(-\Delta)^\alpha U_\mu(x) + U_\mu^p(x) \ge -Ct + t^p|x|^{\tau_0 p}.$$

Then there exists $\delta_2 \in (0, d_0)$ such that

$$(-\Delta)^\alpha U_\mu(x) + U_\mu^p(x) \ge 0, \;\; x \in B_{\delta_2}.$$

Next we consider the domain $\Omega \setminus B_{\delta_2}(0)$. Then, by definition of $U_{\mu,\lambda}$, there exists $C_1 > 0$ such that

$$|(-\Delta)^\alpha V_{\tau_0}| \le C_1 \;\text{ in }\; \Omega \setminus \mathrm{B}_{\delta_2}(0).$$

Then there exists $\mu_1 > 1$ such that for $\mu \ge \mu_1$, it has

$$(-\Delta)^\alpha U_\mu(x) + U_\mu^p(x) \ge \mu - tC_1 \ge 0, \quad x \in \Omega \setminus B_{\delta_2}(0).$$

Then U_{μ_1} is a super-solution of (1.1).

2. Sub-solution. We observe that

$$(-\Delta)^\alpha W_\mu(x) = t(-\Delta)^\alpha V_{\tau_0}(x) - \mu(-\Delta)^\alpha V_{\tau_1}(x) - \mu^2, \quad x \in \Omega \setminus \{0\}.$$

By Corollary 2.1 part (ii) and (iii), for $x \in B_{\delta_1}$, it follows that

$$(-\Delta)^\alpha W_\mu(x) + |W_\mu|^{p-1}W_\mu(x) \;\; \le \;\; Ct - \frac{\mu}{C}|x|^{\tau_1 - 2\alpha} + t^p|x|^{\tau_0 p},$$

where $C > 1$. Here the inequality above we used that for any $a, b \geq 0$,

$$|a-b|^{p-1}(a-b) \leq a^p.$$

Then for $\mu \geq 2Ct^p$ and $\tau_1 - 2\alpha < \tau_0 p$, there exists $\delta_2 > 0$ such that

$$(-\Delta)^\alpha W_\mu(x) + |W_\mu|^{p-1}W_\mu(x) \leq 0, \;\; x \in B_{\delta_2}.$$

Since $(-\Delta)^\alpha V_\tau + V_\tau^p$ is continuous in $\Omega \setminus \{0\}$, then there exists $C_2 > 0$ such that

$$|(-\Delta)^\alpha V_{\tau_0}| + V_{\tau_0}^p \leq C_2, \quad x \in \Omega \setminus B_{\delta_2}(0)$$

and

$$|(-\Delta)^\alpha V_{\tau_1}| \leq C_2, \quad x \in \Omega \setminus B_{\delta_2}(0).$$

Then, there exists $\mu_2 \geq 2Ct^p$ such that for $\mu \geq \mu_2$, we have

$$\begin{aligned} (-\Delta)^\alpha W_\mu(x) + |W_\mu|^{p-1}W_\mu(x) &\leq C_2 t + \mu C_2 + C_2^p t^p - \mu^2 \\ &\leq 0, \quad x \in \Omega \setminus B_{\delta_2}(0). \end{aligned}$$

As a consequence, W_{μ_2} is a sub-solution of (1.1).

Since $\mu_2, \mu_1 > 0$ and $\bar{V}, V_{\tau_0}, V_{\tau_1} > 0$ in $\Omega \setminus \{0\}$, then

$$U_{\mu_1} > W_{\mu_2} \;\text{ in } \Omega \setminus \{0\} \;\text{ and }\; \mathrm{U}_{\mu_1} = \mathrm{W}_{\mu_2} = 0 \text{ in } \Omega^c. \tag{3.4}$$

Then by Lemma 2.2, there exists solution u of (1.1) satisfies (1.8). The proof is complete. □

Proof of Remark 1.2. For any given $t > 0$, we define

$$U_{\mu,\lambda}(x) := tV_{\tau_0}(x) - \mu V_{\tau_1}(x) + \lambda \bar{V}(x)$$

and

$$W_\mu(x) := tV_{\tau_0}(x) - \mu V_{\tau_1}(x) - \mu^2 \bar{V}(x),$$

where $\mu, \lambda > 0$, $\tau_0 = 2\alpha - N$, $\tau_1 = \tau_0 p + 2\alpha$, V_τ is defined in (2.3), and $\bar{V}$ is the solution of (3.2). By $\frac{2\alpha}{N-2\alpha} < p < \frac{N}{N-2\alpha}$, we have that

$$-N + 2\alpha < \tau_1 < 0.$$

We construct super-solution and sub-solution of (1.1) under the hypotheses of Remark 1.2 by adjusting the positive parameters μ and λ.

1. Super-solution. By the definition of $U_{\mu,\lambda}$, it has

$$(-\Delta)^\alpha U_{\mu,\lambda}(x) = t(-\Delta)^\alpha V_{\tau_0}(x) - \mu(-\Delta)^\alpha V_{\tau_1}(x) + \lambda, \quad x \in \Omega \setminus \{0\}.$$

By Corollary 2.1 part (ii) and (iii), for $x \in B_{\delta_1}$, it follows that

$$(-\Delta)^\alpha U_{\mu,\lambda}(x) + U^p_{\mu,\lambda}(x) \ge -Ct - C\mu|x|^{\tau_1-2\alpha} + t^p|x|^{\tau_0 p}.$$

Then letting $\mu = t^p/(2C)$ and there exists $\delta_2 \in (0,\delta_1)$ such that

$$(-\Delta)^\alpha U_{\mu,\lambda}(x) + U^p_{\mu,\lambda}(x) \ge 0, \ \ x \in B_{\delta_2}.$$

Next we consider the domain $\Omega \setminus B_{\delta_2}(0)$. Then, by definition of $U_{\mu,\lambda}$, there exists $C_1 > 0$ such that

$$|(-\Delta)^\alpha V_\tau| \le C_1 \ \text{ in } \ \Omega \setminus \mathrm{B}_{\delta_2}(0),$$

for $\tau = \tau_0, \tau_1$. Then for $\mu = t^p/(2C)$, there exists $\lambda_1 > 1$ such that for $\lambda \ge \lambda_1$, it has

$$\begin{aligned}(-\Delta)^\alpha U_{\mu,\lambda}(x) + |U_{\mu,\lambda}|^{p-1}U_{\mu,\lambda}(x) &\ge \lambda - \mu C_1 - tC_1 - \mu^p C_1^p \\ &\ge 0, \quad x \in \Omega \setminus B_{\delta_2}(0).\end{aligned}$$

Then for $\lambda = \lambda_1 > 1$ and $\mu = \mu_1 = t^p/2$, we have that U_{μ_1,λ_1} is a super-solution of (1.1).

2. Sub-solution. We observe that

$$(-\Delta)^\alpha W_\mu(x) = t(-\Delta)^\alpha V_{\tau_0}(x) - \mu(-\Delta)^\alpha V_{\tau_1}(x) - \mu^2, \quad x \in \Omega \setminus \{0\}.$$

By Corollary 2.1 part (ii) and (iii), for $x \in B_{\delta_1}$, it follows that

$$(-\Delta)^\alpha W_\mu(x) + |W_\mu|^{p-1}W_\mu(x) \ \le \ Ct - \frac{\mu}{C}|x|^{\tau_1-2\alpha} + t^p|x|^{\tau_0 p},$$

where $C > 1$. Here the inequality above we used that for any $a, b \ge 0$,

$$|a-b|^{p-1}(a-b) \le a^p.$$

Then for $\mu \ge 2Ct^p$, there exists $\delta_2 > 0$ such that

$$(-\Delta)^\alpha W_\mu(x) + |W_\mu|^{p-1}W_\mu(x) \le 0, \ \ x \in B_{\delta_2}.$$

Since $(-\Delta)^\alpha V_\tau + V_\tau^p$ is continuous in $\Omega \setminus \{0\}$, then there exists $C_2 > 0$ such that

$$|(-\Delta)^\alpha V_{\tau_0}| + V^p_{\tau_0} \le C_2, \quad x \in \Omega \setminus B_{\delta_2}(0)$$

and

$$|(-\Delta)^\alpha V_{\tau_1}| \le C_2, \quad x \in \Omega \setminus B_{\delta_2}(0).$$

Then, there exists $\mu_2 \geq 2Ct^p$ such that for $\mu \geq \mu_2$, we have

$$\begin{aligned}(-\Delta)^\alpha W_\mu(x) + |W_\mu|^{p-1} W_\mu(x) &\leq C_2 t + \mu C_2 + C_2^p t^p - \mu^2 \\ &\leq 0, \quad x \in \Omega \setminus B_{\delta_2}(0).\end{aligned}$$

As a consequence, W_{μ_2} is a sub-solution of (1.1).

Since $\mu_2 > \mu_1 > 0$ and $\bar{V}, V_{\tau_0}, V_{\tau_1} > 0$ in $\Omega \setminus \{0\}$, then

$$U_{\mu_1,\lambda_1} > W_{\mu_2} \ \text{ in } \Omega \setminus \{0\} \ \text{ and } \ \mathrm{U}_{\mu_1,\lambda_1} = \mathrm{W}_{\mu_2} = 0 \text{ in } \Omega^c. \tag{3.5}$$

Then by Lemma 2.2, there exists solution u of (1.1) satisfies (1.8). The proof is complete. $\square$

The proof of Theorem 1.1 part (ii) with $\frac{2\alpha}{N-2\alpha} < p < \frac{N}{N-2\alpha}$ follows the proof of Remark 1.2.

4 Proof of the uniqueness

In this section, we prove the uniqueness in Theorem 1.1 part (i) by contradiction. Let u and v be two solutions of problem (1.1) satisfying (1.6). We observe that, u and v are positive in $\Omega \setminus \{0\}$ and there exists $C_0 \geq 1$ such that

$$\frac{1}{C_0} \leq v(x)|x|^{-\tau}, \ u(x)|x|^{-\tau} \leq C_0, \quad \forall x \in B_{d_0}, \tag{4.1}$$

where, we recall, $B_{d_0} = B_{d_0}(0) \setminus \{0\}$, $d_0 = \frac{1}{3} dist(0, \partial\Omega)$ and in whole this section, $\tau = -\frac{2\alpha}{p-1}$ of Theorem 1.1 part (i). We denote

$$\mathcal{A} = \{x \in B_{d_0} |\ u(x) > v(x)\}. \tag{4.2}$$

It is easy to see that $\mathcal{A}$ is open and $\mathcal{A} \subset \Omega$.

Theorem 4.1 *Under the hypotheses of Theorem 1.1 part (i), we have*

$$\mathcal{A} = \emptyset.$$

To overcome the difficulty caused by the nonlocal character, we introduce the following lemmas to prove Theorem 4.1. We denote

$$g(x) = \begin{cases} (1 - |x|^2)^3, & x \in B_1(0), \\ 0, & x \in B_1^c(0). \end{cases}$$

Since g is C^2 in $\mathbb{R}^N$, then there exists $\bar{C} > 0$ such that

$$(-\Delta)^\alpha g(x) \leq \bar{C}, \quad x \in B_1(0).$$

Then it is obvious to see that

Lemma 4.1 *Let $V = g/\bar{C}$ in $\mathbb{R}^N$, where $g(x)$ and $\bar{C} > 0$ defined above, then*

$$(-\Delta)^\alpha V(x) \le 1$$

and

$$V(0) = \max_{x \in \mathbb{R}^N} V(x). \tag{4.3}$$

Lemma 4.2 *Under the hypotheses of Theorem 1.1part (i), if*

$$\mathcal{A}_{k,M} := \{x \in \mathbb{R}^N \setminus \{0\} \mid u(x) - kv(x) > M\} \ne \emptyset,$$

for $k > 1$ and $M \ge 0$. Then,

$$0 \in \partial \mathcal{A}_{k,M}. \tag{4.4}$$

Proof. If (4.4) is not true, there exist $\bar{r} > 0$ such that

$$\mathcal{A}_{k,M} \subset \Omega \setminus B_{\bar{r}}(0).$$

Then there exists $\bar{x} \in \Omega \setminus B_{\bar{r}}(0)$ such that

$$u(\bar{x}) - kv(\bar{x}) - M = \max_{x \in \mathbb{R}^N \setminus \{0\}} (u - kv)(x) - M > 0,$$

which follows by $\mathcal{A}_{k,M} \ne \emptyset$. Then, by Lemma 2.1, we have

$$(-\Delta)^\alpha (u - kv)(\bar{x}) \ge 0,$$

which is impossible with

$$\begin{aligned} (-\Delta)^\alpha (u - kv)(\bar{x}) &= -u^p(\bar{x}) + kv^p(\bar{x}) \\ &\le -(k^p - k)v^p(x_0) - M^p \\ &< 0. \end{aligned}$$

We finish the proof. $\square$

By the definition of $\mathcal{A}_{k,M}$ for any $M_1 \ge M_2 \ge 0$, we have that $\mathcal{A}_{k,M_1} \subset \mathcal{A}_{k,M_2}$. For notation convenient, we denote that $\mathcal{A}_k = \mathcal{A}_{k,0}$.

Lemma 4.3 *Under the hypotheses of Theorem 1.1 part (i), if*

$$\mathcal{A}_k \ne \emptyset,$$

where $k > 1$ and $\mathcal{A}_k$ is given above. Then

$$\lim_{r \to 0} \sup_{|x| = r} (u - kv)(x) = +\infty. \tag{4.5}$$

Proof. If not, we have $\bar{M} := \sup_{x\in\mathbb{R}^N\setminus\{0\}}(u-kv)(x) < +\infty$. We see that $\bar{M} > 0$ and there doesn't exist point $\bar{x}$ achieving the supreme of $u-kv$ in $\Omega\setminus\{0\}$. Indeed, if not, we can get a contradiction as in the proof of Lemma 4.2.

By Lemma 4.2, $\mathcal{A}_k$ verifies (4.4). Let $x_0\in\mathcal{A}_k$ chosen later and $r=|x_0|/4$. In the following, we will consider the function

$$w_k = u - kv \quad \text{in} \quad \mathbb{R}^{\mathrm{N}}\setminus\{0\}.$$

Under the hypotheses of Theorem 1.1 part (i), for all $x\in B_r(x_0)\cap\mathcal{A}_k$,

$$(-\Delta)^\alpha w_k(x) = -u^p(x) + kv^p(x), \tag{4.6}$$

then we have that

$$(-\Delta)^\alpha w_k \le -K_1 r^{\tau-2\alpha} \quad \text{in} \quad \mathrm{B_r(x_0)}\cap\mathcal{A}_\mathrm{k}. \tag{4.7}$$

where $\tau p = -\frac{2\alpha p}{p-1} = \tau - 2\alpha$ and $K_1 = C_0^{-p}(k^p - k) > 0$ with C_0 is from (4.1).

We define

$$w(x) = \frac{2\bar{M}}{V(0)} V(r(x-x_0)), \quad x\in\mathbb{R}^N,$$

where V is given in Lemma 4.1, then we see that

$$w(x_0) = \max_{x\in\mathbb{R}^N} w(x) = 2\bar{M} \tag{4.8}$$

and

$$(-\Delta)^\alpha w \le \frac{2\bar{M}}{V(0)} r^{-2\alpha} \quad \text{in} \quad \mathrm{B_r(x_0)}. \tag{4.9}$$

Let $x_0\in\mathcal{A}_k$ close enough to 0 such that

$$\frac{2\bar{M}}{V(0)} \le K_1 r^\tau.$$

Combining (4.7) with (4.9), we have that

$$(-\Delta)^\alpha (w_k + w)(x) \le 0, \quad x\in B_r(x_0)\cap\mathcal{A}_k.$$

By Lemma 2.1 and $w_k(x_0) > 0$, $w = 0$ in $B_r^c(x_0)$, then we have

$$\begin{aligned}
w(x_0) &< w_k(x_0) + w(x_0) \\
&\le \sup_{x\in B_r(x_0)\cap\mathcal{A}_k}(w_k + w)(x) \\
&\le \sup_{x\in (B_r(x_0)\cap\mathcal{A}_k)^c}(w_k + w)(x) \\
&= \max\{\sup_{x\in B_r^c(x_0)} w_k(x), \sup_{x\in B_r(x_0)\cap\mathcal{A}_k^c}(w_k + w)(x)\} \\
&\le \max\{\bar{M}, \sup_{x\in B_r(x_0)\cap\mathcal{A}_k^c}(w_k + w)(x)\}.
\end{aligned} \tag{4.10}$$

We first see the contradiction in case of $\sup_{x\in B_r(x_0)\cap\mathcal{A}_k^c}(w_k+w) > \bar{M}$. By $w_k \le 0$ in $\mathcal{A}_k^c$, we have

$$\sup_{x\in B_r(x_0)\cap\mathcal{A}_k^c}(w_k+w)(x) \le \sup_{x\in B_r(x_0)\cap\mathcal{A}_k^c} w(x),$$

which together with (4.10), we have

$$w(x_0) < \sup_{x\in B_r(x_0)\cap\mathcal{A}_k^c} w(x) \le \sup_{x\in\mathbb{R}^N} w(x) = w(x_0),$$

which is impossible.

We finally see the contradiction in case of $\sup_{x\in B_r(x_0)\cap\mathcal{A}_k^c}(w_k+w) \le \bar{M}$. By (4.10), we have

$$w(x_0) < \bar{M}.$$

which is impossible with (4.8). We finish the proof. $\square$

Remark 4.1 *It is clear that*

$$\mathcal{A}_k \cap B_{d_0} \ne \emptyset.$$

Remark 4.2 *Let* $m_k(t) = \max_{|x|=t} w_k(x)$*, which is continuous in* $(0,+\infty)$ *and*

$$m_k(t) = 0, \quad \forall\ t \ge diam(\Omega).$$

Lemma 4.3 is equivalent to say: if there exist $t_0 > 0$ *such that*

$$m_k(t_0) > 0.$$

Then,

$$\lim_{t\to 0^+} m_k(t) = +\infty.$$

Moreover, let $m_0 = \max_{t\in[t_0,\infty)} m_k(t) > 0$*, then for any* $C > m_0$*, there exists* $t_C \in (0,t_0)$ *such that*

$$m_k(t_C) = C \quad \text{and} \quad \mathrm{m_k(t) \le C \text{ for all } t \in [t_C,\infty)}.$$

Directly by the results of Lemma 4.3, we have

Corollary 4.1 *If* $\mathcal{A}_k \ne \emptyset$ *with* $k > 1$*, then* $\mathcal{A}_{k,M} \ne \emptyset$ *for any* $M \ge 0$*.*

Lemma 4.4 *Let* $x_0 \in \mathcal{A}_k \cap B_{d_0}$, $r = |x_0|/4$ *and*

$$Q_n = \{z \in B_{\frac{r}{n}} \mid w_k(z) > M_n\}, \quad n \in \mathbb{N}$$

with $M_n = \max_{\Omega \setminus B_{\frac{r}{n}}(0)} w_k(x)$, *then there exist* $C_n > 0$ $(n \geq 1)$ *independent of* x_0 *and* k, *such that*

$$\lim_{n \to +\infty} C_n = 0 \tag{4.11}$$

and

$$\int_{Q_n} \frac{w_k(z) - M_n}{|z - x|^{N+2\alpha}} dz \leq C_n r^{\tau - 2\alpha}, \quad \forall x \in B_r(x_0). \tag{4.12}$$

Proof. By $v \geq 0$ in $\mathbb{R}^N \setminus \{0\}$, $M_n \geq 0$ and (4.1), we have

$$w_k(z) - M_n \leq u(z) \leq C_0 |z|^\tau, \quad z \in B_{d_0}.$$

For $x \in B_r(x_0)$ with $r = |x_0|/4$ and $z \in Q_n$, we have

$$|x - z| \geq |x| - |z| \geq 3r - \frac{r}{n} > r.$$

Together with $Q_n \subset B_{\frac{r}{n}} \subset B_{d_0}$, we have

$$\begin{aligned}
\int_{Q_n} \frac{w_k(z) - M_n}{|z - x|^{N+2\alpha}} dz &\leq \int_{Q_n} \frac{u(z)}{|z - x|^{N+2\alpha}} dz \\
&\leq C_0 r^{-N-2\alpha} \int_{B_{\frac{r}{n}}} |z|^\tau dz \\
&\leq C r^{-N-2\alpha} \int_0^{\frac{r}{n}} t^{\tau+N-1} dt \\
&\leq \frac{C}{n^{N+\tau}} r^{\tau - 2\alpha}.
\end{aligned}$$

Let $C_n = \frac{C}{n^{N+\tau}}$, then $\lim_{n \to +\infty} = 0$. The proof is complete. $\square$

Now we give the proof of Theorem 4.1 as follows:

Proof of Theorem 4.1. $\mathcal{A}$ is defined in (4.2). If the conclusion of Theorem 4.1 under hypothesis (i) in Theorem 1.1 isn't true, then $\mathcal{A} \neq \emptyset$.

Let $\bar{x} \in \mathcal{A}$ and $k_0 \in (1, \frac{u(\bar{x})}{v(\bar{x})})$. For example, $k_0 = \frac{u(\bar{x}) + v(\bar{x})}{2v(\bar{x})}$. We observe that $\bar{x} \in \mathcal{A}_{k_0}$. By Corollary 4.1, $\mathcal{A}_{k_0,1}$ is open and nonempty. By Lemma 4.2, we have that

$$0 \in \partial \mathcal{A}_{k_0,1}. \tag{4.13}$$

By using Remark 4.2, there exists $x_0 \in \mathcal{A}_{k_0,1} \cap B_{d_0}$ such that

$$u(x_0) - k_0 v(x_0) = \max_{x \in \Omega \setminus B_{4r}(0)} (u - k_0 v)(x),$$

where $r = |x_0|/4$. We recall that $w_{k_0} = u - k_0 v$, then, by (4.1), for all $x \in B_r(x_0) \cap \mathcal{A}_{k_0,1}$, we have

$$\begin{aligned}
(-\Delta)^\alpha w_{k_0}(x) &= -u^p(x) + k_0 v^p(x) \\
&\le -(k_0^p - k_0) v^p(x) \\
&\le -C_0^{-p}(k_0^p - k_0)|x|^{\tau p} \\
&\le -C_0^{-p}(k_0^p - k_0)(|x_0| - r)^{\tau p} \\
&=: -K_1 r^{\tau - 2\alpha},
\end{aligned}$$

where $\tau = -\frac{2\alpha}{p-1}$, $K_1 = 3^{\tau - 2\alpha} C_0^{-p}(k_0^p - k_0) > 0$ and C_0 is from (4.1). Then we have that

$$(-\Delta)^\alpha w_{k_0} \le -K_1 r^{\tau - 2\alpha} \quad \text{in } \mathrm{B_r(x_0)} \cap \mathcal{A}_{\mathrm{k_0,1}}, \tag{4.14}$$

We redefine

$$w(x) = \frac{K_1 r^\tau}{2} V(r(x - x_0))$$

for $x \in \mathbb{R}^N$, where V is given in Lemma 4.1, then we see that

$$(-\Delta)^\alpha w \le \frac{K_1 r^{\tau - 2\alpha}}{2} \quad \text{in } \mathrm{B_r(x_0)}, \tag{4.15}$$

Combining with (4.14) and (4.15), we have that

$$(-\Delta)^\alpha (w_{k_0} + w)(x) \le -\frac{K_1 r^{\tau - 2\alpha}}{2}, \quad x \in B_r(x_0) \cap \mathcal{A}_{k_0,1}. \tag{4.16}$$

Let

$$M_n := \max_{x \in \overline{B_{5r} \setminus B_{\frac{r}{n}}}} w_{k_0}(x),$$

for $n \ge 1$, we have $x_0 \in B_{5r} \setminus B_{\frac{r}{n}}$, then

$$M_n \ge w_{k_0}(x_0) = \max_{x \in \Omega \setminus B_{4r}(0)} (u - k_0 v)(x). \tag{4.17}$$

We denote that

$$Q_n = \{ z \in B_{\frac{r}{n}} \mid w_{k_0}(z) > M_n \},\ n \in \mathbb{N}$$

and

$$\bar{w}_n(x) = \begin{cases} M_n, & \text{if} \quad \mathrm{x \in Q_n}, \\ (w_{k_0} + w)(x), & \text{if not}. \end{cases} \tag{4.18}$$

By Lemma 4.4, then there exists $n_0 > 1$ such that

$$C_{n_0} \leq \frac{K_1}{2},$$

which, together with (4.16), (4.12), we obtain

$$\begin{aligned} (-\Delta)^\alpha \bar{w}_{n_0}(x) &= (-\Delta)^\alpha (w_{k_0} + w)(x) + \int_{Q_{n_0}} \frac{w_{k_0}(z) - M_{n_0}}{|z-x|^{N+2\alpha}} dz \\ &\leq -\frac{K_1}{2} r^{\tau - 2\alpha} + C_{n_0} r^{\tau - 2\alpha} \\ &\leq 0, \quad x \in B_r(x_0) \cap \mathcal{A}_{k_0,1}. \end{aligned}$$

By Lemma 2.1 and $w_{k_0}(x_0) > 1$, $x_0 \in B_r(x_0) \cap \mathcal{A}_{k_0,1}$, $w = 0$ in $B_r^c(x_0)$, then we have

$$\begin{aligned} w(x_0) + 1 &< w_{k_0}(x_0) + w(x_0) = \bar{w}_{n_0}(x_0) \\ &\leq \sup_{x \in B_r(x_0) \cap \mathcal{A}_{k_0,1}} \bar{w}_{n_0}(x) \\ &\leq \sup_{x \in (B_r(x_0) \cap \mathcal{A}_{k_0,1})^c} \bar{w}_{n_0}(x) \\ &= \max\{\sup_{x \in B_r^c(x_0)} \bar{w}_{n_0}(x), \sup_{x \in B_r(x_0) \cap \mathcal{A}_{k_0,1}^c} \bar{w}_{n_0}(x)\} \\ &\leq \max\{M_{n_0}, \sup_{x \in B_r(x_0) \cap \mathcal{A}_{k_0,1}^c} (w_{k_0} + w)(x)\}. \end{aligned} \tag{4.19}$$

We first claim that $\sup_{x \in B_r(x_0) \cap \mathcal{A}_{k_0,1}^c} (w_{k_0} + w) \leq M_{n_0}$. If not, by $w_{k_0} \leq 1$ in $\mathcal{A}_{k_0,1}^c$, we have

$$\sup_{x \in B_r(x_0) \cap \mathcal{A}_{k_0,1}^c} (w_{k_0} + w)(x) \leq \sup_{x \in B_r(x_0) \cap \mathcal{A}_{k_0,1}^c} w(x) + 1,$$

which together with (4.19), we have

$$w(x_0) + 1 < \sup_{x \in B_r(x_0) \cap \mathcal{A}_{k_0,1}^c} w(x) + 1 \leq \sup_{x \in \mathbb{R}^N} w(x) + 1 = w(x_0) + 1,$$

which is impossible. Then we have that

$$w_{k_0}(x_0) + w(x_0) \leq \sup_{x \in B_r(x_0) \cap \mathcal{A}_{k_0,1}} (w_{k_0} + w)(x) \leq M_{n_0}. \tag{4.20}$$

Since $\bar{\Omega} \setminus B_{\frac{r}{n_0}}(0)$ is compact and (4.17), then there exists $x_1 \in \overline{B_{5r} \setminus B_{\frac{r}{n_0}}}$ such that

$$w_{k_0}(x_1) = M_{n_0}.$$

Together with (4.20), we have

$$\frac{K_1 V(0)}{2} r^\tau = w(x_0) < w_{k_0}(x_0) + w(x_0) \le M_{n_0} = w_{k_0}(x_1). \tag{4.21}$$

By (4.1) and $\frac{r}{n_0} \le |x_1| \le 5r$, we have

$$\frac{K_1 V(0)}{2} \frac{v(x_1)}{5^\tau C_0} \le \frac{K_1 V(0)}{2} r^\tau \le w_{k_0}(x_1) = u(x_1) - k_0 v(x_1),$$

which implies that

$$u(x_1) > (1 + c_0) k_0 v(x_1), \tag{4.22}$$

where

$$c_0 = \frac{3^{\tau - 2\alpha}(k_0^{p-1} - 1)V(0)}{2C_0^2 n_0^{-\tau}} > 0.$$

Now we repeat the process above initiating by x_1. We know that K_1 is increasing with k_0, which is replaced by $k_1 = (1+c_0)k_0$ and n_0 is independent of electing x_0, so we can keep our first choosing of n_0, then we have $x_2 \in \mathcal{A}$ such that

$$u(x_2) > (1 + c_1) k_1 v(x_2) > (1 + c_0)^2 k_0 v(x_2),$$

since

$$c_1 = \frac{3^{\tau - 2\alpha}(k_1^{p-1} - 1)V(0)}{2C_0^2 n_0^{-\tau}} > c_0.$$

Proceeding inductively, we can find a sequence $\{x_m\} \subset \mathcal{A}$ such that

$$u(x_m) > (1 + c_0)^m k_0 v(x_m),$$

which contradicts (4.1). $\square$

With the help of Theorem 4.1, we can prove Theorem 1.1.

Proof the uniqueness in part (i) **of Theorem 1.1.** By $\mathcal{A} = \emptyset$ in Theorem 4.1, then

$$u \le v \ \text{ in } \ \mathrm{B}_{\mathrm{d}_0}.$$

By using Theorem 4.1 in domain $\{x \in B_{d_0} \mid u(x) < v(x)\}$, we see that

$$u \equiv v \ \text{ in } \ \mathrm{B}_{\mathrm{d}_0}.$$

Let $\tilde{w} := u - v$ in $\mathbb{R}^N \setminus \{0\}$.

We first prove $\tilde{w} \geq 0$ in $\mathbb{R}^N \setminus \{0\}$. If not, there exists some point $\bar{x} \in \Omega \setminus B_{d_0}(0)$ such that

$$\tilde{w}(\bar{x}) = \min_{x \in \mathbb{R}^N \setminus \{0\}} \tilde{w}(x) < 0.$$

We observe, on the one hand, that

$$(-\Delta)^\alpha \tilde{w}(\bar{x}) < 0. \tag{4.23}$$

On the other hand, we have that

$$(-\Delta)^\alpha \tilde{w}(\bar{x}) = -u^p(\bar{x}) + v^p(\bar{x}) > 0,$$

which is impossible with (4.23). By the same way, we get $\tilde{w} \leq 0$ in $\mathbb{R}^N \setminus \{0\}$. Then we have that $u \equiv v$ in $\mathbb{R}^N \setminus \{0\}$. We complete the proof. $\square$

5 Nonexistence

In this section, we focus on the nonexistence of classical solutions under the hypotheses of Theorem 1.1 part (iii). The idea of the proof is as following: if there is a solution u for (1.1) such that (1.9) holds for some $\tau \in (-N, 0) \setminus \{2\alpha - N, -\frac{2\alpha}{p-1}\}$, there exists some constants $C_2 \geq C_1 > 0$ such that

$$C_1 = \liminf_{x \to 0} u(x)|x|^{-\tau} \leq \limsup_{x \to 0} u(x)|x|^{-\tau} = C_2.$$

We will find two sub solutions (or both super solutions) U_1 and U_2 such that

$$\lim_{x \to 0} U_1(x) = \frac{C_1}{2}, \quad \lim_{x \to 0} U_2(x) = 2C_2.$$

By using Proposition 5.1 below, we will get a contradiction. Therefore there is no solution under assumption of Theorem 1.1 part (iii).

Proposition 5.1 *Under the hypotheses of Theorem 1.1 part (iii), we suppose that U_1 and U_2 are both sub solutions (or both super solutions) of (1.1) and satisfy that $U_1 = U_2 = 0$ in Ω^c and*

$$\begin{aligned} 0 &< \liminf_{x \to 0} U_1(x)|x|^{-\tau} \leq \limsup_{x \to 0} U_1(x)|x|^{-\tau} \\ &< \liminf_{x \to 0} U_2(x)|x|^{-\tau} \leq \limsup_{x \to 0} U_2(x)|x|^{-\tau} < +\infty, \end{aligned}$$

for some $\tau \in (-N, 0)$. For the case $\tau p > \tau - 2\alpha$, we assume more that
(i) in the case that U_1, U_2 are sub solutions, there exist $C > 0$ and $\bar{\delta} > 0$,

$$(-\Delta)^\alpha U_2(x) \leq -C|x|^{\tau - 2\alpha}, \quad x \in B_{\bar{\delta}}; \tag{5.1}$$

or
(ii) in the case that U_1, U_2 are super solutions, there exist $C > 0$ and $\bar{\delta} > 0$,

$$(-\Delta)^\alpha U_1(x) \geq C|x|^{\tau-2\alpha}, \quad x \in B_{\bar{\delta}}. \tag{5.2}$$

Then there doesn't exist any solution u of (1.1) such that

$$\limsup_{x\to 0} \frac{U_1(x)}{u(x)} < 1 < \liminf_{x\to 0} \frac{U_2(x)}{u(x)}. \tag{5.3}$$

Proof. Here we only prove the case that U_1 and U_2 are sub solutions of (1.1) and the other case could be done similarly. We prove it by contradiction. Assume that there exists a solution u for (1.1) satisfying (5.3). We observe that Lemma 4.2 and Lemma 4.4 hold in $\{x \in \Omega \mid u(x) - kU_1(x) > 0\}$ for any $k > 1$ and Lemma 4.3 holds for $\{x \in \Omega \mid u(x) - kU_1(x) > 1\}$.

Denote $\mathcal{C}_0 = \{x \in \mathbb{R}^N \setminus \{0\} \mid U_2(x) > u(x) > U_1(x) > 1\}$, which is open and nonempty by (5.3). By our hypothesis on U_1, U_2 and (5.3), there exists $C_0 > 1$ such that

$$\frac{1}{C_0} \leq U_1(x)|x|^{-\tau} < u(x)|x|^{-\tau} < U_2(x)|x|^{-\tau} \leq C_0, \quad x \in \mathcal{C}_0. \tag{5.4}$$

Let $\bar{x} \in \mathcal{C}_0$ and $k_0 \in (1, \frac{U_2(\bar{x})}{u(\bar{x})})$. We denote

$$\mathcal{C}_{k_0} := \{x \in \mathbb{R}^N \setminus \{0\} \mid U_2(x) - k_0 u(x) > 1\}$$

which, by Lemma 4.3, is open and nonempty. By Lemma 4.2, we have that

$$0 \in \partial \mathcal{C}_{k_0}.$$

By using Remark 4.2, there exists $x_0 \in \mathcal{C}_{k_0}$ such that

$$u(x_0) - k_0 v(x_0) = \max_{x \in \Omega \setminus B_{4r}(0)} (u - k_0 v)(x),$$

where $r = |x_0|/4$. Let $w_{k_0} = u - k_0 U_1$. In the case of $\tau p \leq \tau - 2\alpha$, by (5.4) we have

$$\begin{aligned}
(-\Delta)^\alpha w_{k_0}(x) &\leq -u^p(x) + k_0 U_1^p(x) \\
&\leq -(k_0^p - k_0) U_1^p(x) \\
&\leq -C_0^{-p}(k_0^p - k_0)(|x_0| - r)^{\tau p} \\
&=: -K_1 r^{\tau - 2\alpha}, \quad x \in B_r(x_0) \cap \mathcal{C}_{k_0},
\end{aligned}$$

where $K_1 = 3^{\tau-2\alpha}C_0^{-p}(k_0^p - k_0) > 0$ and C_0 is from (5.4). In the case of $\tau p > \tau - 2\alpha$, by (5.4) and (5.1) we have

$$\begin{aligned}(-\Delta)^\alpha w_{k_0}(x) &\le -u^p(x) - Ck_0|x|^{\tau-2\alpha} \\ &\le -Ck_0|x|^{\tau-2\alpha}, \quad x \in B_r(x_0) \cap \mathcal{C}_{k_0}.\end{aligned}$$

Proceeding as the Proof of Theorem 4.1, we find a sequence $\{x_m\} \subset \mathcal{C}_0$ such that $u(x_m) > (1+k_1)^m k_0 U_1(x_m)$ for a certain constant $k_1 > 0$, which contradicts (5.4). Then there is no solution of (1.1) satisfying (5.3). $\square$

Now we are in the position to prove Theorem 1.1 part (iii).

Proof of Theorem 1.1 (iii)**.** With the help of Corollary 2.1, for any given $t_1 > t_2 > 0$, we construct two sub solutions (or both super solutions) U_1 and U_2 of (1.1) such that

$$\lim_{x\to 0} U_1(x)|x|^{-\tau} = t_1, \quad \lim_{x\to 0} U_2(x)|x|^{-\tau} = t_2.$$

Then we use Proposition 5.1, we can get there is no solution of (1.1).

We will prove the nonexistence results in 3 cases.

Case 1: $\tau \in (-N, -N+2\alpha)$ and $\tau p > \tau - 2\alpha$. Denote that

$$W_{\mu,t} = tV_\tau - \mu\bar{V} \quad \text{in } \mathbb{R}^{\mathrm{N}} \setminus \{0\},$$

where $t, \mu > 0$, V_τ is defined in (2.3) and $\bar{V}$ is the solution of (3.2). By Corollary 2.1(i), for $x \in B_{\delta_1}$, we have

$$(-\Delta)^\alpha W_{\mu,t}(x) + |W_{\mu,t}|^{p-1}W_{\mu,t}(x) \le -\frac{t}{C}|x|^{\tau-2\alpha} + t^p|x|^{\tau p}.$$

For any fixed $t > 0$, there exists $\delta_2 \in (0, \delta_1]$, for all $\mu \ge 0$, we get

$$(-\Delta)^\alpha W_{\mu,t}(x) + |W_{\mu,t}|^{p-1}W_{\mu,t}(x) \le 0, \quad x \in B_{\delta_2}. \tag{5.5}$$

To consider $x \in \Omega \setminus B_{\delta_2}(0)$, in fact, $(-\Delta)^\alpha V_\tau$ is bounded in $\Omega \setminus B_{\delta_2}(0)$ and

$$(-\Delta)^\alpha W_{\mu,t}(x) + |W_{\mu,t}|^{p-1}W_{\mu,t}(x) \le C(t + t^p) - \mu, \quad x \in \Omega \setminus B_{\delta_2}(0).$$

For given $t > 0$, there exists $\mu(t) > 0$ such that

$$(-\Delta)^\alpha W_{\mu(t),t}(x) + |W_{\mu,t}|^{p-1}W_{\mu,t}(x) \le 0, \quad x \in \Omega \setminus B_{\delta_2}(0). \tag{5.6}$$

Combining with (5.5) and (5.6), we have that for any $t > 0$, there exists $\mu(t) > 0$ such that

$$(-\Delta)^\alpha W_{\mu(t),t}(x) + |W_{\mu(t),t}|^{p-1}W_{\mu(t),t}(x) \le 0, \quad x \in \Omega \setminus \{0\}.$$

For given $t_1 > t_2 > 0$, there exist $\mu(t_1) > 0$ and $\mu(t_2) > 0$ such that

$$t_2 = \lim_{x \to 0} W_{\mu(t_2),t_2}(x)|x|^{-\tau} < \lim_{x \to 0} W_{\mu(t_1),t_1}(x)|x|^{-\tau} = t_1.$$

Using Proposition 5.1 with both sub solutions $W_{\mu(t_1),t_1}$ and $W_{\mu(t_2),t_2}$, there isn't any solution u of (1.1) satisfying (1.9).

Case 2: $\tau \in (-N, -N + 2\alpha)$ and $\tau p < \tau - 2\alpha$. We denote that

$$U_{\mu,t} = tV_\tau + \mu\bar{V} \quad \text{in } \mathbb{R}^{\mathrm{N}} \setminus \{0\},$$

where $t, \mu > 0$. We know that $U_{\mu,t} > 0$ in Ω. By Corollary 2.1 (i), for $x \in B_{\delta_1}$,

$$(-\Delta)^\alpha U_{\mu,t}(x) + U^p_{\mu,t}(x) \geq -Ct|x|^{\tau-2\alpha} + t^p|x|^{\tau p},$$

for some $C > 0$. For any fixed $t > 0$, there exists $\delta_2 \in (0, \delta_1]$, for all $\mu \geq 0$, we have

$$(-\Delta)^\alpha U_{\mu,t}(x) + U^p_{\mu,t}(x) \geq 0, \quad x \in B_{\delta_2}. \tag{5.7}$$

To consider $x \in \Omega \setminus B_{\delta_2}(0)$, in fact, $(-\Delta)^\alpha V_\tau$ is bounded in $\Omega \setminus B_{\delta_2}(0)$ and

$$(-\Delta)^\alpha U_{\mu,t}(x) + U^p_{\mu,t}(x) \geq -Ct + \mu, \quad x \in \Omega \setminus B_{\delta_2}(0).$$

For any given $t > 0$, there exists $\mu(t) > 0$ such that

$$(-\Delta)^\alpha U_{\mu(t),t}(x) + U^p_{\mu(t),t}(x) \geq 0, \quad x \in \Omega \setminus B_{\delta_2}(0). \tag{5.8}$$

Combining with (5.11) and (5.12), we have that for any $t > 0$, there exists $\mu(t) > 0$ such that

$$(-\Delta)^\alpha U_{\mu(t),t}(x) + U^p_{\mu(t),t}(x) \geq 0, \quad x \in \Omega \setminus \{0\}.$$

For given $t_1 > t_2 > 0$, there exist $\mu(t_1) > 0$ and $\mu(t_2) > 0$ such that

$$t_2 = \lim_{x \to 0} U_{\mu(t_2),t_2}(x)|x|^{-\tau} < \lim_{x \to 0} U_{\mu(t_1),t_1}(x)|x|^{-\tau} = t_1,$$

Using Proposition 5.1 with both super solutions $U_{\mu(t_1),t_1}$ and $U_{\mu(t_2),t_2}$, there isn't any solution of (1.1) satisfying (1.9).

Case 3: $\tau \in (-N + 2\alpha, 0)$. By Corollary 2.1$(ii)$, there exists $\delta_1 > 0$ such that

$$(-\Delta)^\alpha V_\tau(x) > 0, \quad x \in B_{\delta_1}. \tag{5.9}$$

Since V_τ is C^2 in Ω, then there exists $C > 0$ such that

$$|(-\Delta)^\alpha V_\tau(x)| \leq C, \quad x \in \Omega \setminus B_{\delta_1}(0). \tag{5.10}$$

Let $\bar{U} := V_\tau + C\bar{V}$, then we have $\bar{U} > 0$ in Ω and

$$(-\Delta)^\alpha \bar{U} \geq 0 \ \text{ in } \ \Omega.$$

Then, we have that $t\bar{U}$ is super solution of (1.1) for any $t > 0$. Using Proposition 5.1, there isn't any solution of (1.1) satisfying (1.9). The proof is complete. $\square$

Proof of Remark 1.3. Since $p \geq \frac{N}{N-2\alpha}$, we have that $-\frac{2\alpha}{p-1} \leq -N$. So by Theorem 1.1, it is only left to prove the case that $\tau = \tau_0 = 2\alpha - N$. We denote that

$$U_{\mu,t} = tV_{\tau_0} + \mu\bar{V} \quad \text{in } \mathbb{R}^{\mathrm{N}} \setminus \{0\},$$

where $t, \mu > 0$. We know that $U_{\mu,t} > 0$ in Ω. By Corollary 2.1 (iii), for $x \in B_{\delta_1}$,

$$(-\Delta)^\alpha U_{\mu,t}(x) + U^p_{\mu,t}(x) \geq -Ct + t^p|x|^{\tau_0 p},$$

for some $C > 0$. For any fixed $t > 0$, there exists $\delta_2 \in (0, \delta_1]$, for all $\mu \geq 0$, we have

$$(-\Delta)^\alpha U_{\mu,t}(x) + U^p_{\mu,t}(x) \geq 0, \quad x \in B_{\delta_2}. \tag{5.11}$$

To consider $x \in \Omega \setminus B_{\delta_2}(0)$, in fact, $(-\Delta)^\alpha V_\tau$ is bounded in $\Omega \setminus B_{\delta_2}(0)$ and

$$(-\Delta)^\alpha U_{\mu,t}(x) + U^p_{\mu,t}(x) \geq -Ct + \mu, \quad x \in \Omega \setminus B_{\delta_2}(0).$$

For any given $t > 0$, there exists $\mu(t) > 0$ such that

$$(-\Delta)^\alpha U_{\mu(t),t}(x) + U^p_{\mu(t),t}(x) \geq 0, \quad x \in \Omega \setminus B_{\delta_2}(0). \tag{5.12}$$

Combining with (5.11) and (5.12), we have that for any $t > 0$, there exists $\mu(t) > 0$ such that

$$(-\Delta)^\alpha U_{\mu(t),t}(x) + U^p_{\mu(t),t}(x) \geq 0, \quad x \in \Omega \setminus \{0\}.$$

For given $t_1 > t_2 > 0$, there exist $\mu(t_1) > 0$ and $\mu(t_2) > 0$ such that

$$t_2 = \lim_{x\to 0} U_{\mu(t_2),t_2}(x)|x|^{-\tau_0} < \lim_{x\to 0} U_{\mu(t_1),t_1}(x)|x|^{-\tau_0} = t_1,$$

Using Proposition 5.1 with both super solutions $U_{\mu(t_1),t_1}$ and $U_{\mu(t_2),t_2}$, there isn't any solution of (1.1) satisfying (1.9). The proof is complete. $\square$

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